Ex 6 Lös

$$\begin{cases} xy' + 2y = \dfrac{\ln x}{x} \\ y(1) = 2 \end{cases}$$

$$xy' + 2y = \frac{\ln x}{x} \iff y' + \frac{2}{x}y = \frac{\ln x}{x^2}$$

Mult. ekv m. den integrerande faktorn $e^{G(x)} = e^{2\ln|x|} \overset{x>0}{=} e^{2\ln x} = e^{\ln x^2} = x^2$:

$$\underbrace{y'\cdot x^2 + y\cdot 2x}_{= D(y\cdot x^2)} = \ln x \iff y\cdot x^2 = \int \ln x \, dx = \int \underset{f}{1}\cdot \underset{g}{\ln x} \, dx = \underset{F}{x}\,\underset{g}{\ln x} - \int \underset{F}{x}\cdot \underset{g'}{\frac{1}{x}} \, dx$$

$$= x\ln x - x + C$$

Allm. lsg: $y(x) = \dfrac{\ln x}{x} - \dfrac{1}{x} + \dfrac{C}{x^2}$

$$y(1) = \frac{\ln 1}{1} - \frac{1}{1} + \frac{C}{1} = C - 1 = 2 \iff C = 3$$

$\therefore$ Sökt lösning: $\underline{y(x) = \dfrac{\ln x}{x} - \dfrac{1}{x} + \dfrac{3}{x^2}}$.

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Ex 7 LÖ

$$\begin{cases} xy' - 2y + \dfrac{x^4}{1+x^2} = 0, & x > 0 \\ y(1) = 0 \end{cases}$$

$$xy' - 2y + \frac{x^2}{1+x^2} = 0 \iff y' - \frac{2}{x}y = -\frac{x^3}{1+x^2}$$

Mult. ekv m. int. fakt. $e^{G(x)} = e^{-2\ln|x|} \underset{x>0}{\overset{\uparrow}{=}} e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$:

$$y' \cdot \frac{1}{x^2} + y \cdot \left(-\frac{2}{x^3}\right) = D\left(y \cdot \frac{1}{x^2}\right) = -\frac{x}{1+x^2}$$

$$\iff y \cdot \frac{1}{x^2} = \int -\frac{x}{1+x^2}\,dx = -\frac{1}{2}\int \frac{\overset{f'}{2x}}{\underset{f}{1+x^2}}\,dx = -\frac{1}{2}\ln|1+x^2| + C = -\frac{1}{2}\ln(1+x^2) + C$$

$$\iff y(x) = \frac{x^2}{2}\left(C - \ln(1+x^2)\right) \leftarrow \text{Allm. lsg.}$$

$$y(1) = \frac{1}{2}(C - \ln 2) = 0 \iff C = \ln 2$$

$\therefore$ Sökt lsg: $\underline{y(x) = \frac{x^2}{2}\left(\ln 2 - \ln(1+x^2)\right)}$.

Ex 10 Lös

$$\begin{cases} y' = y^2(1+xe^{-x}) \\ y(0) = 1 \end{cases}$$

$$y' = y^2(1+xe^{-x}) \Leftrightarrow \underbrace{\frac{1}{y^2}}_{g(y)}\underbrace{\frac{dy}{dx}}_{y'} = \underbrace{1+xe^{-x}}_{h(x)} \Leftrightarrow \int \frac{1}{y^2}\,dy = \int (1+xe^{-x})\,dx$$

$$\Leftrightarrow -\frac{1}{y} = \int (1+xe^{-x})\,dx = \int 1\,dx + \int \underset{g}{x}\,\underset{f}{e^{-x}}\,dx = x + \underset{g}{x}\cdot\underset{F}{(-e^{-x})} - \int \underset{g'}{1}\,\underset{F}{(-e^{-x})}\,dx$$

$$= x - xe^{-x} + \int e^{-x}\,dx = x - xe^{-x} - e^{-x} + C = -e^{-x}(1+x) + x + C$$

$$\Leftrightarrow \quad y(x) = \frac{1}{e^{-x}(1+x) - x - C} \quad \leftarrow \text{Allm. lsg}$$

$$y(0) = \frac{1}{e^{0}(1+0) - 0 - C} = \frac{1}{1-C} = 1 \Leftrightarrow C = 0$$

$$\therefore \text{ sökt lsg: } \quad \underline{y(x) = \frac{1}{e^{-x}(1+x) - x}}.$$

Ex 11 Lös

$$\begin{cases} (1+x^2)y' = x^2y^2, \quad x>0 \\ \lim_{x\to 0^+} y(x) = 1 \end{cases}$$

$$y' + g(x)y = h(x)$$

$$\to \quad g(y)\,y' = h(x)$$

$$(1+x^2)y' = x^2y^2 \Leftrightarrow \underbrace{\frac{1}{y^2}}_{g(y)} \underbrace{\frac{dy}{dx}}_{y'} = \underbrace{\frac{x^2}{1+x^2}}_{h(x)} \Leftrightarrow \int \frac{1}{y^2}\,dy = \int \frac{x^2}{1+x^2}\,dx$$

$$\Leftrightarrow -\frac{1}{y} = \int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int\left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \int\left(1 - \frac{1}{1+x^2}\right)dx = x - \arctan x + C$$

$$\Leftrightarrow \quad y(x) = \frac{1}{\arctan x - x - C} \leftarrow \text{Allm. lsg}$$

$$y(x) = \frac{1}{\underbrace{\arctan x}_{\to 0} - \underbrace{x}_{\to 0} - C} \to -\frac{1}{C} = 1 \text{ då } x \to 0^+ \Leftrightarrow C = -1$$

$\therefore$ Sökt. lsg: $\underline{y(x) = \dfrac{1}{\arctan x - x + 1}}$.

Ex Lös

$$\begin{cases} xy' - y + \dfrac{1}{1+x^2} = 0, & x > 0 \\ y(1) = 1 \end{cases}$$

$\rightarrow$ $y' + g(x)y = h(x)$

$g(y)y' = h(x)$

$$xy' - y + \frac{1}{1+x^2} = 0 \Leftrightarrow y' \underbrace{- \frac{1}{x}}_{g(x)} y = \underbrace{- \frac{1}{x(1+x^2)}}_{h(x)}$$

Mult. ekv med den integrerande faktorn $e^{G(x)} = e^{-\ln|x|} \underset{x>0}{=} e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$:

$$\underbrace{y' \cdot \frac{1}{x} + y \cdot \left(-\frac{1}{x^2}\right)}_{= D\left(y \cdot \frac{1}{x}\right)} = -\frac{1}{x^2(1+x^2)}$$

$$\Leftrightarrow y \cdot \frac{1}{x} = -\int \frac{1}{x^2(1+x^2)} dx = -\int \frac{1+x^2-x^2}{x^2(1+x^2)} dx = -\int \left(\frac{\cancel{1+x^2}}{x^2\cancel{(1+x^2)}} - \frac{\cancel{x^2}}{\cancel{x^2}(1+x^2)} \right) dx$$

$$= \int \left(\frac{1}{1+x^2} - \frac{1}{x^2} \right) dx = \arctan x + \frac{1}{x} + C$$

$$\Leftrightarrow y(x) = x \arctan x + 1 + Cx \leftarrow \text{Allm. lsg.}$$

$$y(1) = 1 \cdot \arctan 1 + 1 + C = \frac{\pi}{4} + 1 + C = 1 \Leftrightarrow C = -\frac{\pi}{4}$$

$\therefore$ Sökt lsg: $\underline{y(x) = x \arctan x + 1 - \frac{\pi x}{4}}$.

Ex Lös

$$\begin{cases} y' = x e^{x-y} \\ y(0) = 0 \end{cases}$$

$$\rightarrow \boxed{\begin{array}{l} y' + g(x) y = h(x) \\ g(y) y' = h(x) \end{array}}$$

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$$y' = x e^{x-y} = x e^{x} \cdot e^{-y} \Leftrightarrow \underbrace{e^{y}}_{g(y)} \cdot y' = \underbrace{x e^{x}}_{h(x)} \Leftrightarrow \int e^{y} dy = \int x e^{x} dx$$

$$y' \leftarrow \frac{dy}{dx}$$

$$\Leftrightarrow \int e^{y} dy = e^{y} = \int \underset{g}{x} \underset{f}{e^{x}} dx = \underset{g}{x} \underset{F}{e^{x}} - \int \underset{g'}{1} \cdot \underset{F}{e^{x}} dx = x e^{x} - e^{x} + C$$

$$\Leftrightarrow y(x) = \ln(x e^{x} - e^{x} + C) \leftarrow \text{Allm. lsg.}$$

$$y(0) = \ln(0 \cdot e^{0} - e^{0} + C) = \ln(C - 1) = 0 \Leftrightarrow C - 1 = 1 \Leftrightarrow C = 2.$$

$\therefore$ Slut lsg: $\underline{y(x) = \ln(e^{x}(x-1) + 2)}$.

Ex Lös $\begin{cases} y' \overbrace{- 2x}^{g(x)} y = 2x^3 \\ y(0) = 0 \end{cases}$ $\rightarrow \begin{cases} y' + g(x) y = h(x) \\ g(y) y' = h(x) \end{cases}$

Mult chv m. den integrerande faktorn $e^{G(x)} = e^{-x^2}$:

$$\underbrace{y' \cdot e^{-x^2} + y e^{-x^2} \cdot (-2x)}_{D(y \cdot e^{-x^2})} = 2x^3 e^{-x^2}$$

$$\Rightarrow \; y \cdot e^{-x^2} = \int 2x^3 e^{-x^2} dx = -\int \underset{g}{x^2} \cdot \underbrace{e^{-x^2}(-2x)}_{f} = -x^2 \cdot e^{-x^2} - \int 2x(-e^{-x^2}) dx$$

$$= -x^2 e^{-x^2} + \int 2x e^{-x^2} dx = -x^2 e^{-x^2} - e^{-x^2} + C$$

$$\Leftrightarrow \; y(x) = -x^2 - 1 + C e^{x^2} \leftarrow \text{Allm lsg}$$

$$y(0) = 0 - 1 + C e^0 = C - 1 = 0 \Leftrightarrow C = 1$$

$\therefore$ Sökt lsg: $\underline{y(x) = e^{x^2} - x^2 - 1}$.

Ex LSS $\begin{cases} y' = 2x(1-y) \\ y(0) = 2 \end{cases}$

Båda! $\rightarrow \begin{cases} y' + g(x)y = h(x) & (1) \\ g(y)y' = h(x) & (2) \end{cases}$

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(1) $y' = 2x(1-y) \Leftrightarrow y' + \underbrace{2x}_{g(x)}y = 2x$

Mult. ekv. m. int. fakt. $e^{G(x)} = e^{x^2}$:

$$\underbrace{y' \cdot e^{x^2} + y\, e^{x^2} \cdot 2x}_{= D(y \cdot e^{x^2})} = 2x\, e^{x^2}$$

$$\Leftrightarrow y \cdot e^{x^2} = \int 2x\, e^{x^2}\, dx = e^{x^2} + C$$

$$\Leftrightarrow y(x) = 1 + C e^{-x^2} \leftarrow \text{Allm. lsg}$$

$$y(0) = 1 + Ce^0 = 1 + C = 2 \Leftrightarrow C = 1$$

$\therefore$ Sökt lsg: $\underline{y(x) = 1 + e^{-x^2}}$.

(2) $y' = 2x(1-y) \underset{\uparrow}{\Leftrightarrow} \overset{g(y)}{\frac{1}{1-y}} \underset{\uparrow}{y'} = \overset{h(x)}{2x}$ *OBS! $y=1$ är en lsg!

Om $y \neq 1$ $\quad$ $\frac{dy}{dx}$

Det är inte den vi söker eftersom $y(0)=2$!

$$\Leftrightarrow \int \frac{1}{1-y}\,dy = \int 2x\,dx$$

$$\Leftrightarrow -\ln|1-y| = x^2 + C \Leftrightarrow \ln|1-y|^{-1} = x^2 + C$$

$$\Leftrightarrow e^{\ln|1-y|^{-1}} = e^{x^2+C} = e^{x^2}\cdot \underbrace{e^{C}}_{=D>0} = De^{x^2}$$

$$\Leftrightarrow \frac{1}{|1-y|} = De^{x^2} \Leftrightarrow \frac{1}{1-y} = \pm De^{x^2} = \underset{\substack{\uparrow \\ E\neq 0}}{E}\cdot e^{x^2}$$

$$\Leftrightarrow 1-y = \frac{1}{Ee^{x^2}} = \frac{1}{E}\cdot e^{-x^2} \Leftrightarrow y(x) = 1 - F\cdot e^{-x^2} \leftarrow \text{Allm lsg!}$$

(* $F=0$ ger $y=1$ ovan)

$$y(0) = 1 - Fe^0 = 2 \Leftrightarrow F = -1 \Leftrightarrow$$

$\therefore$ Sökt lsg: $\quad \underline{y(x) = 1 + e^{-x^2}}$