Ex 7 $\lim_{x\to\infty} \frac{3x^3-2x-8}{x^3+5x^2-6}$ typ "$\frac{\infty}{\infty}$"

$$\frac{3x^3-2x-8}{x^3+5x^2-6} = \frac{\cancel{x^3}\left(3-\frac{2}{x^2}-\frac{8}{x^3}\right)}{\cancel{x^3}\left(1+\frac{5}{x}-\frac{6}{x^3}\right)}$$

$$= \frac{3-\frac{2}{x}-\frac{8}{x^3}}{1+\frac{5}{x}-\frac{6}{x^3}} \to \frac{3-0-0}{1+0-0} = \underline{\underline{3}}$$

då $x \to \infty$

$\frac{2}{x} \to 0$, $\frac{8}{x^3} \to 0$, $\frac{5}{x} \to 0$, $\frac{6}{x^3} \to 0$

Ex 8 $\lim_{x\to\infty} \left(\sqrt{4x^2-x} - 2x\right)$ typ "$\infty-\infty$"

$$\sqrt{4x^2-x} - 2x = \frac{\left(\sqrt{4x^2-x}-2x\right)\left(\sqrt{4x^2-x}+2x\right)}{\sqrt{4x^2-x}+2x}$$

$$= \frac{4x^2-x-4x^2}{\sqrt{4x^2-x}+2x} = \frac{-x}{\sqrt{4x^2-x}+2x}$$

$$= \frac{-x}{\sqrt{x^2\left(4-\frac{1}{x}\right)}+2x} = \frac{-x}{\underbrace{\sqrt{x^2}}_{|x|}\sqrt{4-\frac{1}{x}}+2x}$$

$$= \frac{-x}{|x|\sqrt{4-\frac{1}{x}}+2x} = \frac{-x}{x\sqrt{4-\frac{1}{x}}+2x}$$

$x \to \infty$
$\Rightarrow x > 0 \Rightarrow |x| = x$

$$= \frac{-x}{x\left(\sqrt{4-\frac{1}{x}}+2\right)} = -\frac{1}{\sqrt{4-\frac{1}{x}}+2}$$

$\frac{1}{x} \to 0$

$$\to -\frac{1}{\sqrt{4-0}+2} = \underline{\underline{-\frac{1}{4}}} \text{ då } x \to \infty.$$

Ex 9 $\lim_{x\to\infty}\left(\frac{x+2}{x}\right)^{2x} = ?$

SGV: $\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x} = e$

$$\left(\frac{x+2}{x}\right)^{2x} = \left(\frac{x}{x}+\frac{2}{x}\right)^{2x} = \left(1+\frac{1}{\frac{x}{2}}\right)^{2x} = \Bigg(\underbrace{\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}}}_{\to e}\Bigg)^{4} \to e^4 \text{ då } x\to\infty$$

Ex 10 $\lim_{x\to\infty}\frac{3^{2x}-2^{3x}}{x^{1000}} = ?$ $\left(\text{typ } \frac{\infty-\infty}{\infty}\right)$

SGV: $\lim_{x\to\infty}\frac{a^x}{x^\alpha} = \infty$ om $a>1$

$$\frac{3^{2x}-2^{3x}}{x^{1000}} = \frac{9^x-8^x}{x^{1000}} = \underbrace{\frac{8^x}{x^{1000}}}_{\to\infty}\Bigg(\underbrace{\left(\frac{9}{8}\right)^x-1}_{\to\infty}\Bigg) \to \infty \text{ då } x\to\infty.$$

Ex 17 $\lim_{x\to 0} \frac{3\sin x - \sin 2x}{\sin 3x} = ?$

SGV: $\frac{\sin x}{x} \to 1$ då $x \to 0$

$$\frac{3\sin x - \sin 2x}{\sin 3x} = \frac{3\sin x}{\sin 3x} - \frac{\sin 2x}{\sin 3x} = 3 \cdot \frac{\sin x}{x} \cdot \frac{x}{\sin 3x} - 2\frac{\sin 2x}{2x} \cdot \frac{3x}{3\sin 3x}$$

$$= \frac{\sin x}{x} \cdot \frac{1}{\frac{\sin 3x}{3x}} - 2 \cdot \frac{\sin 2x}{2x} \cdot \frac{1}{\frac{\sin 3x}{3x}} \cdot \frac{1}{3}$$

$$\to 1 \cdot \frac{1}{1} - 2 \cdot 1 \cdot \frac{1}{1} \cdot \frac{1}{3} = 1 - \frac{2}{3} = \underline{\underline{\frac{1}{3}}} \text{ då } x \to 0.$$

Ex 18 $\lim_{x\to 0} \frac{1-\cos x}{x^2} = ?$ (typ "$\frac{0}{0}$")

SGV: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \frac{1}{1+\underbrace{\cos x}_{\to 1}} \to 1 \cdot 1 \cdot \frac{1}{1+1} = \frac{1}{2} \text{ då } x \to 0$$

Ex 21 $\lim_{x\to 0} \frac{\arctan x}{x}$

Sätt $y = \arctan x \Leftrightarrow x = \tan y$, $x \to 0 \Leftrightarrow y \to 0$.

$$\frac{\arctan x}{x} = \frac{y}{\tan y} = \frac{y}{\frac{\sin y}{\cos y}} = \frac{\overbrace{\cos y}^{\to 1}}{\underbrace{\frac{\sin y}{y}}_{\to 1}} \to \frac{1}{1} = 1 \text{ då } y \to 0 \text{ dvs då } x \to 0$$

Ex 19 $\lim_{x\to 0} \frac{\ln(1+x)}{\ln(1+2x)} = ?$

SGV: $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

$$\frac{\ln(1+x)}{\ln(1+2x)} = \frac{\ln(1+x)}{x} \cdot \frac{x}{\ln(1+2x)} = \underbrace{\frac{\ln(1+x)}{x}}_{\to 1} \cdot \frac{1}{\underbrace{\frac{\ln(1+2x)}{2x}}_{\to 1}} \cdot \frac{1}{2}$$

$$\to 1 \cdot \frac{1}{1} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{2}}} \text{ då } x \to 0.$$

Ex 20 $\lim_{x\to \frac{\pi}{2}} \frac{e^{\cos x}-1}{\cos x} = ?$

SGV: $\lim_{x\to 0} \frac{e^{x}-1}{x} = 1$

Sätt $t = \cos x$. $x \to \frac{\pi}{2} \Leftrightarrow t \to 0$

$$\Rightarrow \frac{e^{\cos x}-1}{\cos x} = \frac{e^{t}-1}{t} \to \underline{\underline{1}} \text{ då } t \to 0 \text{ dvs då } x \to \frac{\pi}{2}.$$

Ex 26 Bestäm en asymptoter till kurvan

$$y = f(x) = \frac{2x^3 - x^2}{x^2 - 1} \begin{matrix} \leftarrow T(x) \\ \leftarrow N(x) \end{matrix} \Bigg\} \text{ Rationell funktion!}$$

1) Lodräta?

$N(\pm 1) = 0$, $T(1) = 1 \neq 0$, $T(-1) = -3 \neq 0$

$\therefore$ $y = f(x)$ har de lodräta asymptoterna $\underline{\underline{x = \pm 1}}$.

2) Sneda?

a) Vågräta?

$$f(x) = \frac{2x^3 - x^2}{x^2 - 1} = \frac{\cancel{x^2}(2x - 1)}{\cancel{x^2}(1 - \frac{1}{x^2})} = \frac{2x - 1}{1 - \frac{1}{x^2}} \to \pm\infty \text{ då } x \to \pm\infty$$

$2x - 1 \to \pm\infty$ då $x \to \pm\infty$, $\frac{1}{x^2} \to 0$

$\therefore$ $y = f(x)$ saknar vågrät asymptot.

b) Sneda?

$$k: \quad \frac{f(x)}{x} = \frac{2x^3 - x^2}{x(x^2 - 1)} = \frac{2x^3 - x^2}{x^3 - x} = \frac{2 - \frac{1}{x}}{1 - \frac{1}{x^2}} \to \frac{2 - 0}{1 - 0} = \underline{2} \text{ då } x \to \pm\infty$$

$\frac{1}{x} \to 0$, $\frac{1}{x^2} \to 0$

$$m: \quad f(x) - kx = \frac{2x^3 - x^2}{x^2 - 1} - 2x = \frac{2x^3 - x^2 - 2x(x^2 - 1)}{x^2 - 1} = \frac{-x^2 + 2x}{x^2 - 1}$$

$$= \frac{-1 + \frac{2}{x}}{1 - \frac{1}{x^2}} \to \frac{-1 + 0}{1 - 0} = -1 \text{ då } x \to \pm\infty$$

$\frac{2}{x} \to 0$, $\frac{1}{x^2} \to 0$

$\therefore$ $y = f(x)$ har den sneda asymptoten $\underline{\underline{y = 2x - 1}}$ då $x \to \pm\infty$.

Ex 27 Har $y = f(x) = \sqrt{x}$ någon sned asymptot då $x \to \infty$?

$$k: \quad \frac{f(x)}{x} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \to 0 \text{ då } x \to \infty.$$

$$m: \quad f(x) - kx = \sqrt{x} - 0 \cdot x = \sqrt{x} \to \infty \text{ då } x \to \infty$$

$\therefore$ $y = f(x)$ saknar sned asymptot då $x \to \infty$.