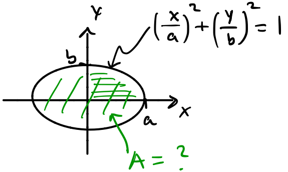
Ex 2   Arean av en ellips



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1 \Leftrightarrow y=f(x)=b\sqrt{1-\left(\frac{x}{a}\right)^2},\ 0\le x\le a.$$

$$\Rightarrow A = 4b\int_0^a \sqrt{1-\left(\frac{x}{a}\right)^2}\,dx = \left\{\begin{array}{l} t=\frac{x}{a} \Leftrightarrow x=at \\ dx = a\,dt \\ x=0 \Leftrightarrow t=0 \\ x=a \Leftrightarrow t=1 \end{array}\right\}$$

$$= 4ab\int_0^1 \sqrt{1-t^2}\,dt = \left\{\begin{array}{l} t=\sin u \\ dt=\cos u\,du \\ t=0 \Leftrightarrow u=0 \\ t=1 \Leftrightarrow u=\frac{\pi}{2} \end{array}\right\} = 4ab\int_0^{\pi/2}\sqrt{1-\sin^2 u}\,\cos u\,du$$

$\cos u > 0$ ty $0 \le x \le \frac{\pi}{2}$

$$= 4ab\int_0^{\pi/2}\sqrt{\cos^2 u}\,\cos u\,du = 4ab\int_0^{\pi/2}|\cos u|\cos u\,du = \int_0^{\pi/2}\cos^2 u\,du$$

$\cos 2u = 2\cos^2 u - 1$

$$= 4ab\int_0^{\pi/2}\frac{\cos 2u+1}{2}\,du = 2ab\left[\frac{\sin 2u}{2}+u\right]_0^{\pi/2} = 2ab\cdot\frac{\pi}{2} = \pi ab \text{ a.e.}$$

Anm: $a=b=c \Rightarrow A_{cirkel} = \pi r^2$ ok!

Ex 3 Längden av kurvan

$$y = f(x) = \frac{2}{3}x\sqrt{x}, \quad 0 \le x \le 1.$$

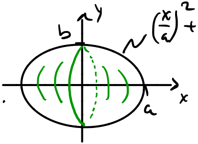
$$\boxed{L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx}$$

$$f(x) = \frac{2}{3}x\sqrt{x} = \frac{2}{3}x^{3/2} \Rightarrow f'(x) = \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = \sqrt{x}$$

$$\Rightarrow L = \int_0^1 \sqrt{1+(\sqrt{x})^2}\,dx = \int_0^1 \sqrt{1+x}\,dx = \int_0^1 (1+x)^{1/2}\,dx = \left[\frac{2(1+x)^{3/2}}{3}\right]_0^1$$

$$= \frac{2}{3}\left((1+1)^{3/2} - (1+0)^{3/2}\right) = \frac{2}{3}\left(2^{3/2} - 1\right) = \frac{2}{3}\left(2\sqrt{2} - 1\right) \text{ L.e.}$$

Ex 4 Volymen av en rotationsellipsoid



$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1 \Leftrightarrow y=f(x)=b\sqrt{1-\left(\frac{x}{a}\right)^2},\ 0\le x\le a.$$

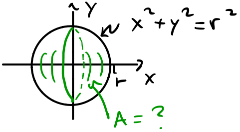
$$\Rightarrow V=2\pi\int_0^a (f(x))^2\,dx = 2\pi b^2\int_0^a \left(1-\left(\frac{x}{a}\right)^2\right)dx$$

$$=2\pi b^2\int_0^a\left(1-\frac{x^2}{a^2}\right)dx = 2\pi b^2\left[x-\frac{x^3}{3a^2}\right]_0^a = 2\pi b^2\left(a-\frac{a^{\cancel{3}\,1}}{3a^{\cancel{2}\,0}}-\left(0-\frac{0^3}{3a^2}\right)\right)$$

$$=2\pi b^2\cdot\frac{2a}{3}=\frac{4\pi b^2 a}{3}$$

Anm: $a=b=r \Rightarrow V_{\text{sfär}}=\frac{4\pi r^3}{3}$ ok!

Ex 6 Arean av en sfär



$f(x)$: $\quad x^2+y^2=r^2 \Leftrightarrow y=f(x)=\sqrt{r^2-x^2}\,, \quad 0\le x\le r$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{r^2-x^2}}\cdot(-2x) = -\frac{x}{\sqrt{r^2-x^2}}$$

$$\Rightarrow \quad \frac{A}{2} = 2\pi\int_0^r \sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\pi\int_0^r \sqrt{r^2-x^2}\sqrt{\frac{r^2-\cancel{x^2}+\cancel{x^2}}{r^2-x^2}}\,dx$$

$$= 2\pi\int_0^r r\,dx = 2\pi\Big[rx\Big]_0^r = 2\pi\cdot r^2$$

$$\therefore \quad A = 4\pi r^2$$