Ex 7 a) $\int_1^2 x^2\,dx = ?$ b) $\int_1^2 e^x\,dx = ?$

a) $$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8-1}{3} = \frac{7}{3}.$$

b) $$\int_1^2 e^x\,dx = \left[e^x\right]_1^2 = e^2 - e^1 = e^2 - e = e(e-1).$$

Ex 7

c) $\displaystyle\int_{-1}^{1} \frac{x}{1+x^2}\,dx = ?$ d) $\displaystyle\int_{-1}^{1} \frac{x^2-1}{1+x^2}\,dx = ?$

c) $$\int_{-1}^{1} \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{-1}^{1} \frac{\overset{f'}{2x}}{\underset{f}{1+x^2}}\,dx = \left[\frac{1}{2}\ln|1+x^2|\right]_{-1}^{1} = 0$$

d) $$\int_{-1}^{1} \frac{x^2-1}{1+x^2}\,dx = \int_{-1}^{1} \frac{x^2-1+1-1}{1+x^2}\,dx = \int_{-1}^{1}\left(\frac{1+x^2}{1+x^2} - \frac{2}{1+x^2}\right)dx$$

$$= \int_{-1}^{1}\left(1 - 2\cdot\frac{1}{1+x^2}\right)dx = \Big[x - 2\arctan x\Big]_{-1}^{1} =$$

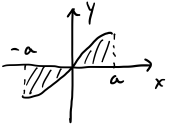
$$= 1 - 2\arctan 1 - \left(-1 - 2\arctan(-1)\right) = 1 - 2\cdot\frac{\pi}{4} - \left(-1 - 2\cdot\left(-\frac{\pi}{4}\right)\right)$$

$$= 2 - \pi.$$

Anm: $f(x) = \dfrac{x}{1+x^2}$ är udda: $f(-x) = -f(x)$

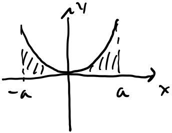
$f(x) = \dfrac{x^2-1}{1+x^2}$ är jämn: $f(-x) = f(x)$

f udda:



$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$$

f jämn:



$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Ex 8 a) $\displaystyle\int_{-3}^{3}\left(|x-2|+|x+1|\right)dx = ?$

$$|x-2|=\begin{cases}x-2, & x\geq 2\\ -(x-2), & x<2\end{cases}, \quad |x+1|=\begin{cases}x+1, & x\geq -1\\ -(x+1), & x<-1\end{cases}$$

$$\Rightarrow \int_{-3}^{3}\left(|x-2|+|x+1|\right)dx = \int_{-3}^{-1}\left(-(x-2)-(x+1)\right)dx+\int_{-1}^{2}\left(-(x-2)+x+1\right)dx$$

$$+\int_{2}^{3}\left(x-2+x+1\right)dx$$

$$=\int_{-3}^{-1}(1-2x)\,dx+\int_{-1}^{2}3\,dx+\int_{2}^{3}(2x-1)\,dx=\left[x-x^2\right]_{-3}^{-1}+\left[3x\right]_{-1}^{2}+\left[x^2-x\right]_{2}^{3}$$

$$=-1-(-1)^2-\left(-3-(-3)^2\right)+3\cdot 2-3\cdot(-1)+3^2-3-\left(2^2-2\right)$$

$$=-1-1+12+6+3+9-3-2=23.$$

Ex 8

b) $$\int_2^4 \frac{4x}{x^3-x^2-x+1}\,dx = ?$$

Faktorisera nämnaren: x^3-x^2-x+1 Gissning $\Rightarrow$ $x=1$ är en rot.

$\Rightarrow$ $x-1$ är en faktor i

Pol. div:

$$\begin{array}{r} x^2-1 \\ x-1 \,\big)\, x^3-x^2-x+1 \\ -(x^3-x^2) \\ \hline -x+1 \\ -(-x+1) \\ \hline 0 \end{array}$$

$$\Rightarrow x^3-x^2-x+1 = (x-1)(x^2-1)$$
$$= (x-1)(x+1)(x-1)$$
$$= (x-1)^2(x+1)$$

PBU:

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + B(x-1)(x+1) + C(x+1)}{(x-1)^2(x+1)}$$

$$= \frac{(A+B)x^2 + (-2A+C)x + A-B+C}{(x-1)^2(x+1)}$$

Ex 8 b)

$$\Leftrightarrow \begin{cases} A+B=0 \\ -2A+C=4 \\ A-B+C=0 \end{cases} \Leftrightarrow \begin{cases} A+B=0 \\ -2A+C=4 \\ 3A-B=-4 \end{cases} \Leftrightarrow \begin{cases} A+B=0 \\ -2A+C=4 \\ 4A=-4 \end{cases} \Leftrightarrow \begin{cases} A=-1 \\ B=1 \\ C=2 \end{cases}$$

$$\Rightarrow \int_2^4 \frac{4x}{x^3-x^2-x+1}\,dx = \int_2^4 \left(-\frac{1}{x+1}+\frac{1}{x-1}+\frac{2}{(x-1)^2}\right)dx = \left[-\ln|x+1|+\ln|x-1|-\frac{2}{x-1}\right]_2^4$$

$$= -\ln|4+1|+\ln|4-1|-\frac{2}{4-1}-\left(-\ln|2+1|+\ln|2-1|-\frac{2}{2-1}\right)$$

$$= -\ln 5+\ln 3-\frac{2}{3}-\left(-\ln 3+\ln 1-2\right) = -\ln 5+2\ln 3+\frac{4}{3}$$

$$= \ln\left(\frac{9}{5}\right)+\frac{4}{3}.$$

Ex 9 a) $\int_1^2 x\ln x\,dx = ?$ b) $\int_0^{\pi^2} \sin\sqrt{x}\,dx = ?$

$$\boxed{\int fg = Fg - \int Fg'}$$

HÖGSKOLAN I HALMSTAD

a) $$\int_1^2 \underset{f}{x}\,\underset{g}{\ln x}\,dx = \left[\underset{F}{\frac{x^2}{2}}\,\underset{g}{\ln x}\right]_1^2 - \int_1^2 \underset{F}{\frac{x^2}{2}}\cdot\underset{g'}{\frac{1}{x}}\,dx = \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^2$$

$$= \frac{2^2\ln 2}{2} - \frac{2^2}{4} - \left(\frac{1^2}{2}\underbrace{\ln 1}_{=0} - \frac{1}{4}\right) = 2\ln 2 - \frac{3}{4}.$$

b) $$\int_0^{\pi^2} \sin\sqrt{x}\,dx = \left\{\begin{array}{l} t=\sqrt{x} \Leftrightarrow x = t^2,\ t\geq 0 \\ dx = 2t\,dt \\ x=0 \Leftrightarrow t=0,\ x=\pi^2 \Leftrightarrow t=\pi \end{array}\right\} = \int_0^{\pi} \underset{f}{\sin t}\cdot\underset{g}{2t}\,dt$$

$$= \left[\underset{F}{-\cos t}\cdot\underset{g}{2t}\right]_0^{\pi} - \int_0^{\pi} \underset{F}{(-\cos t)}\cdot\underset{g'}{2}\,dt = \left[-\cos t\cdot 2t + 2\sin t\right]_0^{\pi}$$

$$= -\underbrace{\cos\pi}_{=-1}\cdot 2\pi + 2\underbrace{\sin\pi}_{=0} - \left(-\underbrace{\cos 0}_{=1}\cdot 2\cdot 0 + 2\underbrace{\sin 0}_{=0}\right) = 2\pi.$$

Ex 9 c) $\displaystyle\int_1^2 (\ln x)^2\,dx = ?$

$$\int_1^2 (\ln x)^2\,dx = \left\{\begin{array}{l} t=\ln x \Leftrightarrow x = e^t \\ dx = e^t\,dt \\ x=1 \Leftrightarrow t=0,\ x=2 \Leftrightarrow t=\ln 2 \end{array}\right\} = \int_0^{\ln 2} \underset{g}{t^2}\cdot \underset{f}{e^t}\,dt$$

$$= \Big[\underset{F}{e^t}\cdot \underset{g}{t^2}\Big]_0^{\ln 2} - \int_0^{\ln 2} \underset{F}{e^t}\cdot \underset{g'}{2t}\,dt$$

$$= \Big[e^t\cdot t^2\Big]_0^{\ln 2} - \left(\overset{f}{\underset{F}{e^t}}\cdot \overset{g}{\underset{g}{2t}} - \int_0^{\ln 2} \underset{F}{e^t}\cdot \underset{g'}{2}\,dt \right)$$

$$= \Big[e^t\cdot t^2 - e^t\cdot 2t + 2e^t\Big]_0^{\ln 2} = \Big[e^t(t^2-2t+2)\Big]_0^{\ln 2}$$

$$= e^{\ln 2}\big((\ln 2)^2 - 2\ln 2 + 2\big) - e^0\big(0^2 - 2\cdot 0 + 2\big)$$

$$= 2\big((\ln 2)^2 - 2\ln 2 + 2\big) - 2 = 2(\ln 2)^2 - 4\ln 2 + 2$$

$$= 2(\ln 2 - 1)^2.$$