Ex 7

$$\boxed{\int fg = Fg - \int Fg'}$$

a) $\int x^2 e^x \, dx = ?$

$$\int \underset{g}{x^2}\,\underset{f}{e^x}\,dx = \underset{F}{e^x}\cdot\underset{g}{x^2} - \int \underset{\underset{f}{F}}{e^x}\cdot\underset{\underset{g}{g'}}{2x}\,dx = e^x\cdot x^2 - \left(\underset{F}{e^x}\cdot\underset{g}{2x} - \int \underset{F}{e^x}\cdot\underset{g'}{2}\,dx\right)$$

$$= e^x x^2 - e^x\cdot 2x - 2e^x + C = e^x(x^2 - 2x + 2) + C$$

b) $\int \arctan x \, dx = ?$

$$\int \arctan x \, dx = \int \underset{f}{1}\cdot\underset{g}{\arctan x}\,dx = \underset{F}{x}\,\underset{g}{\arctan x} - \int \underset{F}{x}\cdot\underset{g'}{\frac{1}{1+x^2}}\,dx$$

$$= x\arctan x - \int \frac{x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{1+x^2}\cdot 2x\,dx$$

$$= x\arctan x - \frac{1}{2}\ln|\underbrace{1+x^2}_{>0}| + C = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

c) $\int e^x \sin x \, dx = ?$

$$I = \int \underset{f}{e^x} \underset{g}{\sin x} \, dx = \underset{F}{e^x} \underset{g}{\sin x} - \int \underset{\substack{F \\ f}}{e^x} \cdot \underset{\substack{g' \\ g}}{\cos x} \, dx$$

$$= e^x \sin x - \left(\underset{F}{e^x} \underset{g}{\cos x} - \int \underset{F}{e^x} \underset{g'}{(-\sin x)} \, dx \right)$$

$$= e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, dx}_{= I}$$

$$\Leftrightarrow \quad I = e^x \sin x - e^x \cos x - I$$

$$\Leftrightarrow \quad 2I = e^x \sin x - e^x \cos x$$

$$\Leftrightarrow \quad I = \int e^x \sin x \, dx = \frac{e^x (\sin x - \cos x)}{2} + C$$

Ex 12 a) $\int \frac{1}{x+\sqrt{x}}\,dx = ?$ b) $\int \cos\sqrt{1+x}\,dx = ?$

HÖGSKOLAN I HALMSTAD

a) $t = \sqrt{x} \Leftrightarrow x = t^2,\ t \geq 0$

$dx = 2t\,dt$

$$\Rightarrow \int \frac{1}{x+\sqrt{x}}\,dx = \int \frac{1}{t^2+t}\cdot 2t\,dt = \int \frac{2\cancel{t}}{\cancel{t}(t+1)}\,dt = 2\int \frac{1}{1+t}\,dt$$

$$= 2\ln|1+t| + C = 2\ln|1+\underbrace{\sqrt{x}}_{\geq 0}| + C = 2\ln(1+\sqrt{x}) + C$$

b) $t = \sqrt{1+x} \Leftrightarrow t^2 = 1+x,\ t \geq 0 \Leftrightarrow x = t^2-1,\ t \geq 0$

$$\boxed{\int fg = Fg - \int Fg'}$$

$dx = 2t\,dt$

$$\Rightarrow \int \cos\sqrt{1+x}\,dx = \int \underset{f}{\cos t}\cdot \underset{g}{2t}\,dt = \underset{F}{\sin t}\cdot \underset{g}{2t} - \int \underset{F}{\sin t}\cdot \underset{g'}{2}\,dt$$

$$= 2t\sin t + 2\cos t + C$$

$$= 2\sqrt{1+x}\,\sin\sqrt{1+x} + 2\cos\sqrt{1+x} + C$$

Ex 21 $\displaystyle\int \frac{2x^4+5x^3+4x^2+x+1}{x^3+2x^2+x}\,dx = ?$ $\leftarrow g(x)$, $\leftarrow h(x)$ $\leftarrow$ Rationell funktion: $f(x) = \dfrac{g(x)}{h(x)}$

1) grad $g >$ grad h: Gör pol.div.!

$$
\begin{array}{r}
2x+1 \\
x^3+2x^2+x \,\big)\, 2x^4+5x^3+4x^2+x+1 \\
-(2x^4+4x^3+2x^2) \\
\hline
x^3+2x^2+x+1 \\
-(x^3+2x^2+x) \\
\hline
1
\end{array}
$$

$\Rightarrow 2x^4+5x^3+4x^2+x+1 = (2x+1)(x^3+2x^2+x)+1$

$\Rightarrow \boxed{f(x) = 2x+1+\dfrac{1}{x^3+2x^2+x}}$

2) $\displaystyle \frac{1}{x^3+2x^2+x} = \frac{1}{x(x^2+2x+1)} = \frac{1}{x(x+1)^2}$

3) PBU:

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2+Bx(x+1)+Cx}{x(x+1)^2}$$

$$= \frac{(A+B)x^2+(2A+B+C)x+A}{x(x+1)^2} \Leftrightarrow \begin{cases} A+B=0 \\ 2A+B+C=0 \\ A=1 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=-1 \\ C=-1 \end{cases}$$

$$\Rightarrow \quad \frac{1}{x(1+x)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$$

4) Kvadrat kompl. behövs ej!

5. Integrera!

$$\int \frac{2x^4+5x^3+4x^2+x+1}{x^3+2x^2+x}\,dx = \int \left(2x+1+\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}\right)dx$$

$$= x^2+x+\ln|x|-\ln|x+1|+\frac{1}{x+1}+C$$

$\underline{\text{Ex}}$ $\int \frac{\ln x}{x^2}\,dx = ?$ $\int fg = Fg - \int Fg'$

HÖGSKOLAN I HALMSTAD

$$\int \frac{\ln x}{x^2}\,dx = \int \frac{t}{e^{2t}} \cdot e^t\,dt = \int \frac{t}{e^t}\,dt = \int \underset{g}{t}\, \underset{F}{e^{-t}}\,dt$$

$$\begin{bmatrix} t = \ln x \Leftrightarrow x = e^t \\ dx = e^t dt \end{bmatrix}$$

$$= \underset{F}{-e^{-t}} \cdot \underset{g}{t} - \int \underset{F}{(-e^{-t})} \cdot \underset{g'}{1} = -e^{-t} \cdot t - e^{-t} \cdot 1 + C = -e^{-t}(t+1) + C$$

$$= -\frac{1}{x}(\ln x + 1) + C$$

Snabbare alt:

$$\int \frac{\ln x}{x^2}\,dx = \int \underset{f}{\frac{1}{x^2}} \cdot \underset{g}{\ln x}\,dx = \underset{F}{-\frac{1}{x}} \cdot \underset{g}{\ln x} - \int \underset{F}{\left(-\frac{1}{x}\right)} \cdot \underset{g'}{\frac{1}{x}}\,dx$$

$$= -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + C$$

$$= -\frac{1}{x}(\ln x + 1) + C$$

Ex $\displaystyle\int \frac{1}{x^{3/2}+x}\,dx = ?$

$$\begin{cases} t=\sqrt{x} \Leftrightarrow x=t^2,\ t\geq 0 \\ dx = 2t\,dt \end{cases}$$

$$\int \frac{1}{x^{3/2}+x}\,dx = \int \frac{1}{x(\sqrt{x}+1)}\,dx \overset{\downarrow}{=} \int \frac{1}{t^2(t+1)}\cdot 2t\,dt = 2\int \frac{1}{t(t+1)}\,dt$$

PBU:

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{A(t+1)+Bt}{t(t+1)} = \frac{(A+B)t+A}{t(t+1)} \Leftrightarrow \begin{cases} A+B=0 \\ A=1 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$\Rightarrow \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\left(\underline{\text{Alt:}} \quad \frac{1}{t(t+1)} = \frac{1+t-t}{t(t+1)} = \frac{\cancel{1+t}}{t\cancel{(t+1)}} - \frac{\cancel{t}}{\cancel{t}(t+1)} = \frac{1}{t} - \frac{1}{t+1} \right)$$

$$\Rightarrow \int \frac{1}{x^{3/2}+x}\,dx = 2\int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt = 2\left(\ln|t| - \ln|t+1|\right) + C$$

$$= 2\left(\ln|\sqrt{x}| - \ln|\sqrt{x}+1|\right) + C = 2\left(\ln\sqrt{x} - \ln(\sqrt{x}+1)\right) + C$$